

Perturbation analysis of multi-asset portfolio optimization with transaction cost

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We employ perturbation analysis technique to study trading strategies for multi-asset portfolio and obtain optimal trading methods for wealth maximization under arbitrary utility functions.

Most financial models that had been studied in the past carry the unrealistic assumption that trading transaction is free. In recent years, the study of portfolio optimization under non-zero transaction cost has finally received its due attention [1, 2]. In the literature, it was generally assumed that there are only one risk-free asset (bond) and one risky asset (stock), albeit with one notable paper by Atkinson and Mokkavesa [2] where multi-asset portfolio optimization is considered. With perturbation method, the authors are able to obtain optimal investment strategy with arbitrary consumption habit. In this paper, we go beyond the consumption habit consideration and generalize the discussion to wealth maximization. Specifically, we consider the strategy that optimizes the wealth, defined according to some arbitrary utility function, after a fixed length of time. The tool we employ to treat this problem is perturbation analysis on the corresponding Hamilton-Bellman-Jacobi (HBJ) equation (for a review, see, e.g., [3]).

We consider a market with investment opportunities on n stocks and a risk free bond. Let $S_i(t)$ be the spot price of stock i at time t , $A_i(t)$ and $B(t)$ be the value of assets invested in stock i , and risk free bonds respectively, and $\Pi(t)$ be the value of the whole portfolio: $\Pi = B + \sum_{i=1}^n A_i$. An investor has a fixed interval $[0, T]$ in which to invest. We assume $S_i(t)$ follows a geometric Brownian motion with growth rate $\mu_i > 0$ and volatility $\sigma_i > 0$. The risk free bonds, B , compounds continuously with risk free rate r . The volatilities σ_i , growth rates μ_i and interest rate r are constants. Cash generated or needed from the purchase or sale of stocks is immediately invested or withdrawn from the risk free bonds. We will now firstly describe the method through the analysis of portfolio optimisation without transaction cost. This problem is easily solved without recourse to perturbation analysis and will serve to familiarise the readers with the usefulness of the HJB equation.

The model is represented by

$$\begin{aligned} dA_i &= \mu_i A_i dt + \sigma_i A_i dX_i, \quad i = 1, \dots, n \\ dB &= rB dt = r(\Pi - \sum_{i=1}^n A_i) dt \end{aligned}$$

$$d\Pi = rB dt + \sum_{i=1}^n \mu_i A_i dt + \sum_{i=1}^n \sigma_i A_i dX_i \quad (1)$$

where X_i , $i = 1, \dots, n$, are standard Brownian motions whose correlations, $-1 \leq \rho_{ij} \leq 1$, are assumed constant. At time $t = 0$, an investor has an amount Π_0 of resources. The problem is to allocate investments over the given time horizon so as to maximise the following expectation value:

$$\mathbf{E} \left\{ F(\Pi(T)) + \int_0^T I(\Pi(t')) dt' \right\}$$

where the functions I and F can represent anything from utility to the year end bonus of a trader. For instance, if we assume that $I = 0$ and $F(\Pi(T)) = \log(\Pi(T))$, then the optimization problem constitutes the Long Term Growth Model and the goal would be to optimise the logarithm of the final wealth.

We restate the above equation in dynamic programming form which will then allow us to transform the problem to the corresponding HBJ equation. To do so, we define the optimal expected value function $J(\Pi, t)$ as

$$J(\Pi, t) = \max_{A_i} \mathbf{E}_t \left\{ F(\Pi) + \int_t^T I(\Pi(t')) dt' \right\} \quad (2)$$

where, to make financial sense, it is assumed that:

$$\frac{\partial I}{\partial \Pi} \geq 0, \quad \frac{\partial^2 I}{\partial \Pi^2} \leq 0 \quad (3)$$

$$\frac{\partial F}{\partial \Pi} \geq 0, \quad \frac{\partial^2 F}{\partial \Pi^2} \leq 0 \quad (4)$$

and one of $\frac{\partial^2 I}{\partial \Pi^2}$ or $\frac{\partial^2 F}{\partial \Pi^2}$ has to be strictly less than 0 [4].

Applying the Bellman Principle and Itô's Lemma to equation (2) the HBJ equations becomes [3]:

$$\begin{aligned} 0 &= \max_{A_1, \dots, A_n} \left[I + \frac{\partial J}{\partial t} + r(\Pi - \sum_{i=1}^n A_i) \frac{\partial J}{\partial \Pi} \right. \\ &\quad \left. + \sum_{i=1}^n \mu_i A_i \frac{\partial J}{\partial \Pi} + \frac{1}{2} \sum_{i,j=1}^n \Omega_{ij} A_i A_j \frac{\partial^2 J}{\partial \Pi^2} \right] \quad (5) \end{aligned}$$

with the boundary condition $J(\Pi, T) = F(\Pi(T))$ and A_i as the control parameters from the perspective of dynamics programming. In the above equation, Ω is the standard covariance matrix. By first diagonalizing the

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correlation matrix and then invest in proportional according to the weights of the corresponding eigenvectors (buying for positive weights and short selling for negative weights), one can eliminate the correlation between the weighted combinations of stocks. Hence we will from now on assume that all the diagonal elements in Ω are zeros. We can rewrite eqn 5 in matrix notation as

$$0 = \max_{\vec{A}} \left[J_t + I + r\Pi J_\Pi + \vec{\mu}^T \vec{A} J_\Pi + \frac{1}{2} \vec{A}^T \Omega \vec{A} J_{\Pi\Pi} \right]. \quad (6)$$

where $\vec{\mu}^T = (\mu_1 - r, \dots, \mu_n - r)$. By differentiating the above with respect to \vec{A} , one obtain as the solution to the Bellman Equation: $\frac{\partial J}{\partial \Pi} \vec{\mu} + \Omega \vec{A} \frac{\partial^2 J}{\partial \Pi^2} = 0$, or equivalently

$$\vec{A} = -\frac{\Omega^{-1} \vec{\mu} \frac{\partial J}{\partial \Pi}}{\frac{\partial^2 J}{\partial \Pi^2}}. \quad (7)$$

Example: the Long Term Growth Model. Apply the multi-asset optimization to the Long Term Growth Model that is to maximize $\mathbf{E}[\log \Pi]$, we obtain

$$J(\Pi, t) = \log \Pi + (r + \frac{1}{2} \vec{\mu}^T \Omega^{-1} \vec{\mu})(T - t). \quad (8)$$

and $\vec{A}^* = \Pi \Omega^{-1} \vec{\mu}$.

We will now include transaction cost to our discussion. As the transaction cost usually amounts to 0.01-1% of the total transaction, the strategy is to expand the value function in terms of the transaction cost and by keeping track of the first few lowest order terms, we will derive the first order correction to the optimal trading strategy under no transaction cost. We note that the accuracy of the analysis could be further improved by going to higher orders in the analysis.

We assume that the transaction cost is proportional to the stock price with the proportionality constants denoted by k , the market model equations are thus represented by

$$\begin{aligned} dB &= rBdt - (1+k)dL_i(t) + (1-k)dM_i(t) \\ &= r(\Pi - \sum_{i=1}^n A_i)dt - (1+k)dL_i(t) + (1-k)dM_i(t) \\ dA_i &= \mu_i A_i dt + dL_i(t) - dM_i(t) + \sigma_i A_i dX_i, \quad i = 1, \dots, n \\ d\Pi &= r\Pi + \sum_{i=1}^n \left(-rA_i dt + \mu_i A_i dt + \sigma_i A_i dX_i \right) \\ &\quad - \sum_{i=1}^n \left(kdL_i(t) + k dM_i(t) \right) \end{aligned} \quad (9)$$

where $L_i(t)$ and $M_i(t)$ represent the cumulative purchase and sale of assets A_i in $[0, T]$. The optimal expected value function $J(\Pi, \vec{A}, t)$ is

$$J(\Pi, \vec{A}, t) = \max_{L_i, M_i} \mathbf{E} \left\{ F(\Pi(T)) + \int_t^T I(\Pi(t')) dt' \right\}. \quad (10)$$

and the corresponding HBJ equation is [3]:

$$\begin{aligned} &\max_{L_i, M_i} \left\{ I + \frac{\partial J}{\partial t} + \sum_{i=1}^n \left(\mu_i A_i + \frac{dL_i}{dt} - \frac{dM_i}{dt} \right) \frac{\partial J}{\partial A_i} \right. \\ &+ \left[r(\Pi - \sum_{i=1}^n A_i) + \sum_{i=1}^n \left(\mu_i A_i - k \frac{dL_i}{dt} - k \frac{dM_i}{dt} \right) \right] \frac{\partial J}{\partial \Pi} \\ &\left. + \sum_{i=1}^n \sigma_i^2 A_i^2 \left(\frac{1}{2} \frac{\partial^2 J}{\partial A_i^2} + \frac{\partial^2 J}{\partial \Pi^2} + \frac{\partial^2 J}{\partial A_i \partial \Pi} \right) \right\} = 0 \end{aligned}$$

where in the case, L_i and M_i are the control parameters from the dynamics programming perspective.

With regard to the above equation, we now consider three separate cases:

Case 1: $\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} < 0$ and $-\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} \geq 0$.

In this case, the maximum is achieved by choosing $dL_i = 0$ and $dM_i = \infty$ which suggests selling at the maximum rate.

Case 2: $\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} \geq 0$ and $-\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} \leq 0$.

In this case, the maximum is achieved by choosing $dL_i = \infty$ and $dM_i = 0$ which suggests buying at the maximum rate.

Case 3: $\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} < 0$ and $-\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi} < 0$.

In this case, the maximum is achieved by choosing $dL_i = 0$ and $dM_i = 0$ which suggests no transaction is needed.

We note that it is not possible to have $\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi}$ and $-\frac{\partial J}{\partial A_i} - k \frac{\partial J}{\partial \Pi}$ be both greater than zero as we assume that J is an increasing function of Π . This can be broadly interpreted as more wealth cannot decrease the value function from the trader's point of view.

With the above consideration, the optimal trading strategy, given t , Π and \vec{A} , can therefore be partitioned into three possible regions: sales, purchase and no-transaction regions. Inside the no-transaction region, $d\vec{L}$ and $d\vec{M}$ are identically zero and hence J satisfies HBJ equation with $k = 0$ (no transaction cost). At the boundary between sales region and no transaction region, we assume that J is continuous and differentiable. The necessity of this assumption is more thoroughly discussed in [5, 6]. Suppose the point (Π, A, t) is at the sales region, when a very small quantity of assets h is sold, the risk-free bond increases by an amount of $h(1-k)$, while the whole portfolio value is reduced by kh . As the value function J must be the same after the sales (the continuity assumption), we have

$$\begin{aligned} J(\Pi + kh, A, t) &= J(\Pi, A - h, t) \\ k \frac{J(\Pi + kh, A, t) - J(\Pi, A, t)}{kh} &= \frac{J(\Pi, A - h, t) - J(\Pi, A, t)}{h} \end{aligned}$$

As $h \rightarrow 0$, the above equation becomes $k \frac{\partial J}{\partial \Pi} = -\frac{\partial J}{\partial A}$. From the above arguments, we know that when the portfolio is in the sales region, the optimal strategy is to sell

stocks until the portfolio is at the no-transaction region boundary, and thus bring the portfolio back into the no-transaction region. In the purchase-no-transaction region, we similarly have $k \frac{\partial J}{\partial \Pi} = \frac{\partial J}{\partial A}$. Due to the optimality assumption [6], we also have the smooth pasting condition at the sales-no-transaction and purchase-no-transaction boundaries. Specifically, we assume that $\frac{\partial J}{\partial A}$ exists and is continuous across the sales-no-transaction boundary. Using the same argument as in the value matching consideration earlier, we have $\frac{\partial J}{\partial A} \Big|_{(\Pi+k h, A, t)} = \frac{\partial J}{\partial A} \Big|_{(\Pi, A-h, t)}$, or equivalently, $\frac{\partial^2 J}{\partial A^2} = -k \frac{\partial^2 J}{\partial A \partial \Pi}$, at the sales-no-transaction boundary; and the equality: $\frac{\partial^2 J}{\partial A^2} = k \frac{\partial^2 J}{\partial A \partial \Pi}$ at the purchase-no-transaction boundary.

To re-cap, at the purchase-no-transaction boundaries for stock i , $i = 1, \dots, n$, J satisfies $k \frac{\partial J}{\partial \Pi} = \frac{\partial J}{\partial A_i}$ and $\frac{\partial^2 J}{\partial A_i^2} = -k \frac{\partial^2 J}{\partial A_i \partial \Pi}$. At the sales-no-transaction boundaries, J satisfies $k \frac{\partial J}{\partial \Pi} = \frac{\partial J}{\partial A_i}$ and $\frac{\partial^2 J}{\partial A_i^2} = k \frac{\partial^2 J}{\partial A_i \partial \Pi}$. These equalities are to be supplemented by the boundary condition at $t = T$: $J(\Pi, \vec{A}, T) = F(\Pi)$.

For $i = 1, \dots, n$, we redefine the A_i coordinate as $A_i = \mathcal{A}_i(\Pi, t) + k^{1/3} \alpha_i$, where \mathcal{A}_i is the value of stock i when the k_i tends to zero, i.e., when the transaction cost goes to zero. We let $H(\Pi, \vec{\alpha}, t) = J(\Pi, \vec{A}, t)$ and we further expand $H(\Pi, \vec{\alpha}, t)$ in powers of $k^{1/3}$ as:

$$H_0(\Pi, \vec{\alpha}, t) + k^{1/3} H_1(\Pi, \vec{\alpha}, t) + k^{2/3} H_2(\Pi, \vec{\alpha}, t) + k H_3(\Pi, \vec{\alpha}, t) + k^{4/3} H_4(\Pi, \vec{\alpha}, t) + \mathcal{O}(k^{5/3}) \quad (11)$$

The reason for expanding H and A_i in powers of $k^{1/3}$ is out of necessity and is previously studied in the literature [1] (for a simple explanation, see [7]).

We will from now on keep track of the expression up to the first non-trivial correction: $\mathcal{O}(k^{5/3})$. The boundary condition at $t = T$ gives the following conditions:

$$H_0(\Pi, \vec{\alpha}, T) = F(\Pi) \quad (12)$$

$$H_m(\Pi, \vec{\alpha}, T) = 0, \quad 1 \leq m \leq 4 \quad (13)$$

By matching the orders of k , the continuity conditions at the sales-no-transaction boundary (corresponds to the + sign in \pm) and at the purchase-no-transaction boundary (corresponds to the - sign in \pm) become:

$$\text{For } 0 \leq m \leq 2, \quad \frac{\partial H_m}{\partial \alpha_i} = 0 \quad (14)$$

$$\frac{\partial H_3}{\partial \alpha_i} \pm \left(- \sum_{j=1}^n \frac{\partial \mathcal{A}_j}{\partial \Pi} \frac{\partial H_0}{\partial \alpha_j} \right) = 0 \quad (15)$$

$$\frac{\partial H_4}{\partial \alpha_i} \pm \frac{\partial H_0}{\partial \Pi} \pm \left(- \sum_{j=1}^n \frac{\partial \mathcal{A}_j}{\partial \Pi} \frac{\partial H_1}{\partial \alpha_j} \right) = 0 \quad (16)$$

and the smooth pasting equations become:

$$\text{For } 0 \leq m \leq 2, \quad \frac{\partial^2 H_m}{\partial \alpha_i^2} = 0 \quad (17)$$

$$\frac{\partial^2 H_3}{\partial \alpha_i^2} \pm \left(- \sum_{j=1}^n \frac{\partial \mathcal{A}_j}{\partial \Pi} \frac{\partial^2 H_0}{\partial \alpha_j^2} \right) = 0 \quad (18)$$

$$\frac{\partial^2 H_4}{\partial \alpha_i^2} \pm \frac{\partial^2 H_0}{\partial \alpha_i \partial \Pi} \pm \left(- \sum_{j=1}^n \frac{\partial \mathcal{A}_j}{\partial \Pi} \frac{\partial^2 H_1}{\partial \alpha_j^2} \right) = 0 \quad (19)$$

At the no-transaction region, after expanding H according to equation (11) and collecting terms of the same order in k , we have the equations below:

$\mathcal{O}(k^{-2/3})$ Equation: $\mathcal{D}H_0 = 0$, where \mathcal{D} is an operator defined as $\sum_{i,j=1}^n \mathcal{D}_{ij} \partial_{\alpha_i \alpha_j}^2$ where \mathcal{D}_{ij} is:

$$\frac{1}{2} \frac{\partial \mathcal{A}_i}{\partial \Pi} \frac{\partial \mathcal{A}_j}{\partial \Pi} \sum_{h=1}^n \sigma_h^2 \mathcal{A}_h^2 + \sigma_i^2 \mathcal{A}_i^2 \left(\frac{1}{2} \delta_{ij} - \frac{\partial \mathcal{A}_i}{\partial \Pi} \right) \quad (20)$$

$\mathcal{O}(k^{-1/3})$ Equation: $\mathcal{D}H_1 = 0$.

$\mathcal{O}(1)$ Equation: $\mathcal{D}H_2 = -\mathcal{M}H_0$, where \mathcal{M} is an operator defined as

$$\partial_t + I + r(\Pi - \sum_{i=1}^n \mathcal{A}_i) \partial_\Pi + \sum_{i=1}^n \left(\mu_i \mathcal{A}_i \partial_\Pi + \frac{1}{2} \sigma_i^2 \mathcal{A}_i^2 \partial_\Pi^2 \right). \quad (21)$$

$\mathcal{O}(k^{1/3})$ Equation: $\mathcal{D}H_3 = -\sum_{i=1}^n \alpha_i \partial_{\mathcal{A}_i} (\mathcal{M}H_0) - \mathcal{M}H_1$.

$\mathcal{O}(k^{2/3})$ Equation: $\mathcal{D}(H_4) = -\frac{1}{2} \sum_{i=1}^n \sigma_i^2 \frac{\partial^2 H_0}{\partial \Pi^2} \alpha_i^2 - \mathcal{M}H_2$.

Combining the $\mathcal{O}(k^{-2/3})$ equation with equations (14) and (17) when $m = 0$, one finds that H_0 is independent of $\vec{\alpha}$. Combining the $\mathcal{O}(k^{-1/3})$ with equations (14) and (17) when $m = 1$ shows that H_1 is independent of $\vec{\alpha}$; Combining the $\mathcal{O}(1)$ equation with equations (14) and (17) when $m = 2$ shows that H_2 is independent of $\vec{\alpha}$. The $\mathcal{O}(k^{1/3})$ equation together with equations (15) and (18) imply that H_3 is independent of $\vec{\alpha}$. Together with boundary condition in equation (13), one finds that H_1 is in fact 0. In other words, by matching the coefficients of the various orders in k , we find that $H_1 = 0$, and H_0, H_2, H_3 are independent of $\vec{\alpha}$.

Without loss of generality, we let α_+ denotes the purchase-no-transaction boundary for α_1 and α_- for the sales-no-transaction boundary. From eqns 16, we know at α_- :

$$\frac{\partial H_4}{\partial \alpha_1} + \frac{\partial H_0}{\partial \Pi} = 0 \quad \text{and} \quad \frac{\partial^2 H_4}{\partial \alpha_1^2} = 0; \quad (22)$$

and at α_+ , we have:

$$\frac{\partial H_4}{\partial \alpha_1} - \frac{\partial H_0}{\partial \Pi} = 0 \quad \text{and} \quad \frac{\partial^2 H_4}{\partial \alpha_1^2} = 0. \quad (23)$$

As we have established that H_2 is independent of $\vec{\alpha}$, with the $\mathcal{O}(k^{2/3})$ equation, we can conclude that H_4 has the

following form in general:

$$H_4 = \sum_{\substack{j_1, \dots, j_n = 0 \\ \text{s.t. } j_1 + \dots + j_n \leq 4}}^4 \hat{h}_{j_1 \dots j_n} \alpha_1^{j_1} \dots \alpha_n^{j_n}, \quad (24)$$

i.e., a multinomial with a total degree of at most four.

Let us concentrate on the boundary for α_1 and redefine the h s by absorbing the relevant α_m for $m > 1$. In other words, the expression for H_4 is simplified to:

$$H_4 = \sum_{j=0}^4 h_1^j \alpha_1^j$$

where h_m^j can be functions of $\alpha_m, m > 1$. We now make the simplifying assumption that $|\alpha_+| = |\alpha_-|$. This is equivalent to saying that the transaction (buy or sell) boundaries are the same distance away from the unperturbed optimal curve. We note that this assumption is proved in the 2-risky-asset case [5].

From the third equality in equations (22) and (23), we know that α_+ and α_- are the roots of

$$0 = 6h_1^4 \alpha_1^2 + 3h_1^3 \alpha_1 + h_1^2. \quad (25)$$

The assumption that $|\alpha_+| = |\alpha_-|$ renders h_1^3 zero. Now, equations (16) implies that α_- and α_+ satisfy respectively:

$$\pm \frac{\partial H_0}{\partial \Pi} = 4h_1^4 \alpha_{\pm}^3 + 2h_1^2 \alpha_{\pm} + h_1^1. \quad (26)$$

Since $\alpha_+ = -\alpha_-$, summing the above equations gives: $h_1^1 = 0$. Substituting eqn 25 into eqn 26, we have

$$\alpha_{\pm}^3 = \pm \frac{1}{16h_1^4} \frac{\partial H_0}{\partial \Pi}. \quad (27)$$

To calculate h_1^4 , we invoke the $\mathcal{O}(k^{2/3})$ equation: By comparing the coefficient of the α_1^2 term on both sides, we find:

$$h_1^4 = -\frac{\sigma_1^2}{24\mathcal{D}_{11}} \frac{\partial^2 H_0}{\partial \Pi^2}. \quad (28)$$

So finally, α_{\pm} can be expressed as:

$$\alpha_{\pm}^3 = \mp \frac{3\mathcal{D}_{11}}{2\sigma_1^2} \frac{\partial H_0}{\partial \Pi} \left(\frac{\partial^2 H_0}{\partial \Pi^2} \right)^{-1} \quad (29)$$

where H_0 is the optimal trading strategy when there is no transaction cost.

As an example, let us consider the Long Term Growth Model where $H_0(\Pi) = \log \Pi$ and $\vec{A} = \Pi \Omega^{-1} \vec{\mu}$ (c.f. eqn 8), the width of the boundary for stock 1 is therefore:

$$\left\{ \frac{3\Pi^3 k}{2\sigma_1^4} \left[\frac{\hat{\mu}_1^2}{2\sigma_1^2} \sum_{j=1}^n \frac{\hat{\mu}_j^2}{\sigma_j^2} + \frac{\hat{\mu}_1^2}{2} - \frac{\hat{\mu}_1^3}{\sigma_1^2} \right] \right\}^{1/3} \quad (30)$$

with similar expressions for other stocks. We note that the first term in the square brackets summarizes the coupling between the different stocks.

In conclusion, we have employed perturbation theory to study multi-asset optimisation for wealth maximization under arbitrary utility function. We believe that our analysis is of interest to other generic stochastic systems.

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